MATH2118 Lecture Notes

Further Engineering Mathematics C

Vector Calculus

Dr Eddie Ly

School of Mathematical & Geospatial Sciences (SMGS)

College of Sciences, Engineering & Health (SEH)

RMIT University, Melbourne, Australia

Summer Course 2015

Contents

1 Introduction

Physical quantities such as temperature, pressure, density, electric field and magnetic field are functions of position in that they vary throughout space (and time). A scalar function defined throughout some region is called a *scalar field*, while a vector function defined throughout some region is called a *vector field*.

2 Gradient and Gradient Operator

The gradient of a scalar function $\phi(x, y, z)$ is defined by

$$\boxed{\operatorname{grad}\phi = \boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k},}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the x, y and z-direction, respectively. Note that $\boldsymbol{\nabla}\phi$, which is often written as $\operatorname{grad}\phi$, is a vector function, and $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are partial derivative operators with respect to the variable x, y and z, respectively. For example, if $f(x, y, z) = x^2yz$, then $\frac{\partial f}{\partial x} = f_x = (2x)yz$; that is, differentiating function f with respect to x only while keeping all other variables fixed.

EXAMPLE

If $\phi(x, y, z) = x^2 + 3xyz - y^2$, determine $\boldsymbol{\nabla}\phi$ at the point $P(1, -2, 5)$.

SOLUTION

Gradient of the scalar function $\phi(x, y, z)$,

$$\begin{aligned}\boldsymbol{\nabla}\phi &= \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k} \\ &= (2x + 3yz)\boldsymbol{i} + (3xz - 2y)\boldsymbol{j} + 3xy\boldsymbol{k}.\end{aligned}$$

Thus, at the point $P(1, -2, 5)$, we have

$$\boldsymbol{\nabla}\phi\big|_P = -28\boldsymbol{i} + 19\boldsymbol{j} - 6\boldsymbol{k}.$$

EXAMPLE

Determine

(1) $\boldsymbol{\nabla} r^3$,

(2) $\boldsymbol{\nabla}(1/r)$ and

(3) $\boldsymbol{\nabla} \ln r$.

SOLUTION

Note that $\boldsymbol{r} = \boldsymbol{r}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2}$.

(1) Firstly, we compute

$$\boldsymbol{\nabla}\left(r^3\right) = \frac{\partial}{\partial x}\left(r^3\right)\boldsymbol{i} + \frac{\partial}{\partial y}\left(r^3\right)\boldsymbol{j} + \frac{\partial}{\partial z}\left(r^3\right)\boldsymbol{k}.$$

Utilising the chain rule, we obtain for the first term on the right side of the above expression,

$$\begin{aligned}\frac{\partial}{\partial x}\left(r^3\right) &= \frac{\partial}{\partial r}\left(r^3\right)\frac{\partial r}{\partial x} \\ &= 3r^2\frac{\partial r}{\partial x}\,.\end{aligned}$$

Since $r^2 = x^2 + y^2 + z^2$, we have upon differentiating both sides (partially) with respect to the variable x (keeping both variables y and z fixed),

$$\begin{aligned}\frac{\partial}{\partial x}\left(r^2\right) &= \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) \\ \Rightarrow\ 2r\frac{\partial r}{\partial x} &= 2x \\ \Rightarrow\quad \frac{\partial r}{\partial x} &= \frac{x}{r}\,.\end{aligned}$$

Alternatively, since $r = \sqrt{x^2 + y^2 + z^2}$, it follows that

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{1/2} \\ &= \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}\frac{\partial}{\partial x}\left(x^2\right) \\ &= \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} \\ &= \frac{x}{r}.\end{aligned}$$

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$. Thus,

$$\begin{aligned}\boldsymbol{\nabla}\left(r^3\right) &= \frac{\partial}{\partial x}\left(r^3\right)\boldsymbol{i} + \frac{\partial}{\partial y}\left(r^3\right)\boldsymbol{j} + \frac{\partial}{\partial z}\left(r^3\right)\boldsymbol{k} \\ &= 3r^2\frac{\partial r}{\partial x}\boldsymbol{i} + 3r^2\frac{\partial r}{\partial y}\boldsymbol{j} + 3r^2\frac{\partial r}{\partial z}\boldsymbol{k}\end{aligned}$$

$$
\begin{aligned}
&= 3r^2\frac{x}{r}\boldsymbol{i} + 3r^2\frac{y}{r}\boldsymbol{j} + 3r^2\frac{z}{r}\boldsymbol{k} \\
&= 3r^2\left(\frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{r}\right) \\
&= 3r^2\,\frac{\boldsymbol{r}}{r} \\
&= 3r\boldsymbol{r} \\
&= 3|\boldsymbol{r}|\boldsymbol{r} \quad \text{since } r = |\boldsymbol{r}|.
\end{aligned}
$$

(2) Using the method similar for Question (1), we obtain

$$
\begin{aligned}
\boldsymbol{\nabla}(1/r) &= \frac{\partial}{\partial x}(1/r)\boldsymbol{i} + \frac{\partial}{\partial y}(1/r)\boldsymbol{j} + \frac{\partial}{\partial z}(1/r)\boldsymbol{k} \\
&= \left(\frac{\partial}{\partial r}\frac{1}{r}\right)\frac{\partial r}{\partial x}\boldsymbol{i} + \left(\frac{\partial}{\partial r}\frac{1}{r}\right)\frac{\partial r}{\partial y}\boldsymbol{j} + \left(\frac{\partial}{\partial r}\frac{1}{r}\right)\frac{\partial r}{\partial z}\boldsymbol{k} \\
&= -\frac{1}{r^2}\frac{x}{r}\boldsymbol{i} - \frac{1}{r^2}\frac{y}{r}\boldsymbol{j} - \frac{1}{r^2}\frac{z}{r}\boldsymbol{k} \\
&= \frac{-(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})}{r^3} \\
&= -\frac{\boldsymbol{r}}{r^3} \\
&= -\frac{1}{r^2}\,\frac{\boldsymbol{r}}{r} \\
&= -\frac{\hat{\boldsymbol{r}}}{|\boldsymbol{r}|^2},
\end{aligned}
$$

where $\hat{\boldsymbol{r}} = \dfrac{\boldsymbol{r}}{|\boldsymbol{r}|}$ is an unit vector of $\boldsymbol{r}$.

(3) Applying the chain rule provides

$$
\begin{aligned}
\boldsymbol{\nabla}(\ln r) &= \frac{\partial}{\partial x}(\ln r)\boldsymbol{i} + \frac{\partial}{\partial y}(\ln r)\boldsymbol{j} + \frac{\partial}{\partial z}(\ln r)\boldsymbol{k} \\
&= \left(\frac{\partial}{\partial r}\ln r\right)\frac{\partial r}{\partial x}\boldsymbol{i} + \left(\frac{\partial}{\partial r}\ln r\right)\frac{\partial r}{\partial y}\boldsymbol{j} + \left(\frac{\partial}{\partial r}\ln r\right)\frac{\partial r}{\partial z}\boldsymbol{k} \\
&= \frac{1}{r}\cdot\frac{x}{r}\boldsymbol{i} + \frac{1}{r}\cdot\frac{y}{r}\boldsymbol{j} + \frac{1}{r}\cdot\frac{z}{r}\boldsymbol{k} \\
&= \frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{r^2} \\
&= \frac{\boldsymbol{r}}{r}\cdot\frac{1}{r} \\
&= \frac{\hat{\boldsymbol{r}}}{|\boldsymbol{r}|}.
\end{aligned}
$$

2.1 Directional Derivative

If $\phi(x, y, z)$ is a scalar function, then

$$\boxed{\frac{d\phi}{du} = \boldsymbol{\nabla}\phi \cdot \hat{\boldsymbol{u}}}$$

is the *directional derivative* of $\phi(x, y, z)$ in the direction specified by the vector $\boldsymbol{u}$. If θ is the angle between $\boldsymbol{\nabla}\phi$ and $\boldsymbol{u}$, we have

$$\begin{aligned}\frac{d\phi}{du} &= \boldsymbol{\nabla}\phi \cdot \hat{\boldsymbol{u}} \\ &= \left|\boldsymbol{\nabla}\phi\right|\left|\hat{\boldsymbol{u}}\right| \cos\theta \\ &= \left|\boldsymbol{\nabla}\phi\right| \cos\theta \quad \text{since } \left|\hat{\boldsymbol{u}}\right| = 1.\end{aligned}$$

In the above expressions, $\hat{\boldsymbol{u}}$ is an unit vector of $\boldsymbol{u}$ and the operator "$\cdot$" represents the dot product operation. Since the maximum value of $\cos\theta$ is one, and occurs when $\theta = 0$, we can conclude that

- At any given point, $\left|\boldsymbol{\nabla}\phi\right|$ is the maximum directional derivative;
- At any given point, the directional derivative is greatest in the direction of $\boldsymbol{\nabla}\phi$ (since $\theta = 0$).

EXAMPLE

If the temperature of a body at the point (x, y, z) is

$$T(x, y, z) = 120 - x^2 + 3xyz - y^2 + 4y,$$

determine the direction in which the temperature increases most rapidly at the point $P(1, 2, 1)$.

SOLUTION

The temperature increases most rapidly in the direction of $\boldsymbol{\nabla}T$. Now,

$$\begin{aligned}\boldsymbol{\nabla}T &= \frac{\partial T}{\partial x}\boldsymbol{i} + \frac{\partial T}{\partial y}\boldsymbol{j} + \frac{\partial T}{\partial z}\boldsymbol{k} \\ &= (-2x + 3yz)\boldsymbol{i} + (3xz - 2y + 4)\boldsymbol{j} + 3xy\boldsymbol{k}, \\ \Rightarrow \quad \boldsymbol{\nabla}T\big|_P &= (4, 3, 6).\end{aligned}$$

3 Vector Fields

A general vector function has the form

$$\boldsymbol{v}(x,y,z) = v_1(x,y,z)\boldsymbol{i} + v_2(x,y,z)\boldsymbol{j} + v_3(x,y,z)\boldsymbol{k}.$$

If $\boldsymbol{v}(x,y,z)$ is defined at each point of a region R, then $\boldsymbol{v}$ is said to form a *vector field over* R. Some examples of vector field are

- motion of a wind or fluid, since a vector (both speed and direction) can be assigned at each point representing the velocity of a particle at the point;
- electric intensity and magnetic intensity are vector functions, which depend on time as well as position;
- laminar flow of blood in an artery, where cylindrical layers of blood flow faster near the centre of the artery.

3.1 Divergence of a Vector Field

Recalling that the gradient of a scalar function ϕ is defined by

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x}\boldsymbol{i} + \frac{\partial \phi}{\partial y}\boldsymbol{j} + \frac{\partial \phi}{\partial z}\boldsymbol{k}.$$

Note that the ordinary derivative $\dfrac{dy}{dx}$ can be written as $\dfrac{d}{dx}(y)$, where $\dfrac{d}{dx}$ is the ordinary derivative operator. In a similar way, $\boldsymbol{\nabla}\phi$ can be interpreted as

$$\boldsymbol{\nabla}\phi = \Bigg(\underbrace{\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}}_{=\boldsymbol{\nabla}}\Bigg)\phi.$$

Here $\boldsymbol{\nabla}$ is called the *gradient operator* (or del operator),

$$\boxed{\boldsymbol{\nabla} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}}$$

and it behaves both like a “vector” and a “differential operator”.

3.2 Divergence

The dot (scalar) product of the gradient operator $\boldsymbol{\nabla}$ and a vector function $\boldsymbol{F}$ is called the *divergence* of $\boldsymbol{F}$ (or div $\boldsymbol{F}$). To be specific, if

$$\boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k},$$

then the divergence of $\boldsymbol{F}$ is defined by

$$\begin{aligned}\operatorname{div}\boldsymbol{F} &= \nabla \cdot \boldsymbol{F} \\ &= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \cdot \big(F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}\big) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\,.\end{aligned}$$

Note that div $\boldsymbol{F}$ produces a <u>scalar function</u> as a result.

EXAMPLE

Determine the divergence of

(1) $\boldsymbol{F} = e^{3x}yz\boldsymbol{i} + x\sin y\boldsymbol{j} + \big(z^2+5\big)\boldsymbol{k}\,;$

(2) $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}\,;$

(3) $\boldsymbol{F} = \dfrac{x\boldsymbol{i} + y\boldsymbol{j}}{x^2+y^2}\,.$

<u>SOLUTION</u>

(1) Using div $\boldsymbol{F} = \boldsymbol{\nabla} \cdot \boldsymbol{F}$ gives

$$\begin{aligned}\operatorname{div}\boldsymbol{F} &= \boldsymbol{\nabla} \cdot \boldsymbol{F} \\ &= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \cdot \Big(e^{3x}yz\boldsymbol{i} + x\sin y\boldsymbol{j} + \big(z^2+5\big)\boldsymbol{k}\Big) \\ &= \frac{\partial}{\partial x}\big(e^{3x}yz\big) + \frac{\partial}{\partial y}\big(x\sin y\big) + \frac{\partial}{\partial z}\big(z^2+5\big) \\ &= 3e^{3x}yz + x\cos y + 2z.\end{aligned}$$

(2) For $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$, we have

$$\begin{aligned}
\operatorname{div}\boldsymbol{F} &= \boldsymbol{\nabla}\cdot\boldsymbol{F}\\
&= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\cdot\big(yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}\big)\\
&= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)\\
&= 0+0+0\\
&= 0.
\end{aligned}$$

The vector function $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$ is said to be *solenoidal*, because it has the property that $\operatorname{div}\boldsymbol{F} = 0$.

(3) Here, $\boldsymbol{F}$ represents the effect of a point source located at the origin in fluid mechanics,

$$\begin{aligned}
\operatorname{div}\boldsymbol{F} &= \boldsymbol{\nabla}\cdot\boldsymbol{F}\\
&= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x}{x^2+y^2}\boldsymbol{i} + \frac{y}{x^2+y^2}\boldsymbol{j} + 0\,\boldsymbol{k}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\\
&= \frac{(1)\big(x^2+y^2\big) - x(2x)}{(x^2+y^2)^2} + \frac{(1)\big(x^2+y^2\big) - y(2y)}{(x^2+y^2)^2}\\
&= \frac{\big(y^2-x^2\big) + \big(x^2-y^2\big)}{(x^2+y^2)^2}\\
&= 0 \quad \text{provided } (x,y) \neq (0,0).
\end{aligned}$$

This $\boldsymbol{F}$ is also solenoidal, except at the origin.

EXAMPLE

Determine the divergence of $\boldsymbol{F} = r^2\boldsymbol{r}$.

SOLUTION

Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $r = |\boldsymbol{r}| = \sqrt{x^2+y^2+z^2}$. Then,

$$\begin{aligned}
\operatorname{div}\boldsymbol{F} &= \boldsymbol{\nabla}\cdot\boldsymbol{F}\\
&= \boldsymbol{\nabla}\cdot\big(r^2\boldsymbol{r}\big)\\
&= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\cdot\big(xr^2\boldsymbol{i} + yr^2\boldsymbol{j} + zr^2\boldsymbol{k}\big)\\
&= \frac{\partial}{\partial x}\Big(x\big(x^2+y^2+z^2\big)\Big) + \frac{\partial}{\partial y}\Big(y\big(x^2+y^2+z^2\big)\Big) + \frac{\partial}{\partial z}\Big(z\big(x^2+y^2+z^2\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \left(x^2+y^2+z^2\right) + x(2x) + \left(x^2+y^2+z^2\right) + y(2y) + \left(x^2+y^2+z^2\right) + z(2z) \\
&= 3\left(x^2+y^2+z^2\right) + 2\left(x^2+y^2+z^2\right) \\
&= 5\left(x^2+y^2+z^2\right) \\
&= 5r^2 \\
&= 5\boldsymbol{r}\cdot\boldsymbol{r}
\end{aligned}
$$

Alternatively, we can find divergence of $\boldsymbol{F} = r^2\boldsymbol{r}$ by first writing $\boldsymbol{F}$ as

$$\boldsymbol{F} = r^2x\boldsymbol{i} + r^2y\boldsymbol{j} + r^2z\boldsymbol{k}.$$

Divergence of $\boldsymbol{F}$:

$$
\begin{aligned}
\operatorname{div}\boldsymbol{F} &= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right)\cdot\left(r^2x\boldsymbol{i} + r^2y\boldsymbol{j} + r^2z\boldsymbol{k}\right) \\
&= \frac{\partial}{\partial x}\left(r^2x\right) + \frac{\partial}{\partial y}\left(r^2y\right) + \frac{\partial}{\partial z}\left(r^2z\right).
\end{aligned}
$$

Appling the product rule to the above relation gives

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(r^2x\right) &= x\frac{\partial}{\partial x}\left(r^2\right) + r^2\frac{\partial}{\partial x}(x) \\
&= x(2r)\frac{\partial r}{\partial x} + r^2(1) \\
&= 2xr\frac{x}{r} + r^2 \quad \text{since } \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{r} \\
&= 2x^2 + r^2.
\end{aligned}
$$

By symmetry, the other terms are

$$\frac{\partial}{\partial y}\left(r^2y\right) = 2y^2 + r^2 \quad \text{and} \quad \frac{\partial}{\partial z}\left(r^2z\right) = 2z^2 + r^2.$$

Hence,

$$
\begin{aligned}
\operatorname{div}\boldsymbol{F} &= \frac{\partial}{\partial x}\left(r^2x\right) + \frac{\partial}{\partial y}\left(r^2y\right) + \frac{\partial}{\partial z}\left(r^2z\right) \\
&= \left(2x^2+r^2\right) + \left(2y^2+r^2\right) + \left(2z^2+r^2\right) \\
&= 2\left(x^2+y^2+z^2\right) + 3r^2 \\
&= 5r^2 \\
&= 5\boldsymbol{r}\cdot\boldsymbol{r}.
\end{aligned}
$$

REMARKS

The last example describes a special case of the "product" rule for the divergence operator. If $\phi(x, y, z)$ is a scalar function and

$$\boldsymbol{F}(x, y, z) = F_1(x, y, z)\boldsymbol{i} + F_2(x, y, z)\boldsymbol{j} + F_3(x, y, z)\boldsymbol{k}$$

is a general vector function, then

$$\begin{aligned}
\operatorname{div} \phi\boldsymbol{F} &= \boldsymbol{\nabla} \cdot \phi\boldsymbol{F} \\
&= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \cdot \left(\phi F_1\boldsymbol{i} + \phi F_2\boldsymbol{j} + \phi F_3\boldsymbol{k}\right) \\
&= \frac{\partial}{\partial x}(\phi F_1) + \frac{\partial}{\partial y}(\phi F_2) + \frac{\partial}{\partial z}(\phi F_3) \\
&= \left(\phi\frac{\partial F_1}{\partial x} + F_1\frac{\partial \phi}{\partial x}\right) + \left(\phi\frac{\partial F_2}{\partial y} + F_2\frac{\partial \phi}{\partial y}\right) + \left(\phi\frac{\partial F_3}{\partial z} + F_3\frac{\partial \phi}{\partial z}\right) \\
&= \underbrace{\left(F_1\frac{\partial \phi}{\partial x} + F_2\frac{\partial \phi}{\partial y} + F_3\frac{\partial \phi}{\partial z}\right)}_{=\boldsymbol{F}\cdot\boldsymbol{\nabla}\phi} + \phi\underbrace{\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)}_{=\phi\boldsymbol{\nabla}\cdot\boldsymbol{F}} \\
&= \boldsymbol{F} \cdot \boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla} \cdot \boldsymbol{F}.
\end{aligned}$$

For example, in the last example, let $\phi = r^2$ (scalar function) and $\boldsymbol{F} = \boldsymbol{r}$. Since the divergence is a derivative operator, rules of differentiation must be obeyed. Thus,

$$\operatorname{div} \phi\boldsymbol{F} \neq \phi \operatorname{div} \boldsymbol{F},$$

except when ϕ is constant (since $\boldsymbol{\nabla}\phi = \boldsymbol{0}$).

3.3 Physical Interpretation

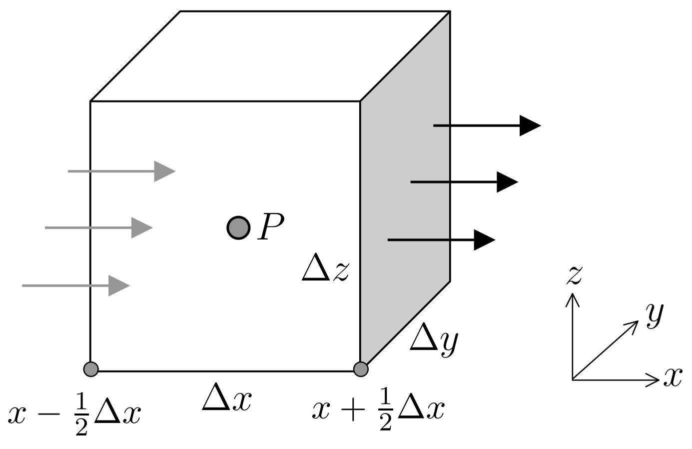
In the modelling of fluid flow, we let

$$\boldsymbol{v} = u(x, y, z)\boldsymbol{i} + v(x, y, z)\boldsymbol{j} + w(x, y, z)\boldsymbol{k}$$

be the steady fluid velocity at any point $P(x, y, z)$. Then, for per unit volume and per unit time at point P,

- $\operatorname{div} \boldsymbol{v}$ represents the *net volume outflow*;
- $\operatorname{div}(\rho\boldsymbol{v})$ represents the *net mass outflow*, where $\rho(x, y, z)$ is the fluid density.

Considering the net mass outflow through a small box of infinitesimal length Δx, Δy and Δz *(control volume)* containing point P in the x-direction as shown below,



we have

$$\begin{aligned}\Delta M_x &\approx M_{x_{\text{OUT}}} - M_{x_{\text{IN}}}\\ &\approx \rho u\Big|_{x+\frac{\Delta x}{2}}(\Delta y\Delta z) - \rho u\Big|_{x-\frac{\Delta x}{2}}(\Delta y\Delta z)\\ &\approx \left(\frac{\rho u\Big|_{x+\frac{\Delta x}{2}} - \rho u\Big|_{x-\frac{\Delta x}{2}}}{\Delta x}\right)(\Delta x\Delta y\Delta z)\\ &\to \frac{\partial}{\partial x}(\rho u)(\Delta x\Delta y\Delta z) \quad \text{as } \Delta x \to 0.\end{aligned}$$

Combining this with the net mass outflow in the y and z-direction, and letting

$$\Delta V = \Delta x\,\Delta y\,\Delta z,$$

we have the total mass outflow per unit time,

$$\begin{aligned}\Delta M &= \Delta M_x + \Delta M_y + \Delta M_z\\ &\to \Bigg(\underbrace{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)}_{=\boldsymbol{\nabla}\cdot(\rho \boldsymbol{v})}\Bigg)\Delta V \quad \text{as } (\Delta x, \Delta y, \Delta z) \to 0.\end{aligned}$$

Thus, the net mass outflow per unit time, per unit volume at P is represented by

$$\begin{aligned}\frac{\Delta M}{\Delta V} &= \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\\ &= \operatorname{div}(\rho\boldsymbol{v}).\end{aligned}$$

In steady case, provided there is no net change in mass at P, that is $\Delta M = 0$, *conservation of mass* requires that

$$\boxed{\operatorname{div}(\rho\boldsymbol{v}) = 0.}$$

If the fluid is *incompressible* (ρ is constant), this relation reduces to

$$\boxed{\operatorname{div}\boldsymbol{v} = 0 \quad \longleftarrow \boldsymbol{v} \text{ is solenoidal for incompressible fluid!}}$$

4 Curl of a Vector Field

The cross (vector) product of $\boldsymbol{\nabla}$ and a vector function $\boldsymbol{F}$ is called the *curl of* $\boldsymbol{F}$ (or $\operatorname{curl}\boldsymbol{F}$). If

$$\boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k},$$

then the curl of $\boldsymbol{F}$ is defined by

$$\begin{aligned}\operatorname{curl}\boldsymbol{F} &= \boldsymbol{\nabla}\times\boldsymbol{F}\\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}\\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}.\end{aligned}$$

Note that $\operatorname{curl}\boldsymbol{F}$ produces a vector function as a result.

▌EXAMPLE

Determine the curl of

(1) $\boldsymbol{F} = 2xy\boldsymbol{i} + yz^2\boldsymbol{j} + x^3z\boldsymbol{k}$;

(2) $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$;

(3) $\boldsymbol{F} = r^2\boldsymbol{r}$.

SOLUTION

(1) Giving $\boldsymbol{F} = 2xy\boldsymbol{i} + yz^2\boldsymbol{j} + x^3z\boldsymbol{k}$, curl of $\boldsymbol{F}$ is

$$\begin{aligned}\operatorname{curl}\boldsymbol{F} &= \boldsymbol{\nabla}\times\boldsymbol{F}\\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xy & yz^2 & x^3z\end{vmatrix}\\ &= \boldsymbol{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz^2 & x^3z\end{vmatrix} - \boldsymbol{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ 2xy & x^3z\end{vmatrix} + \boldsymbol{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ 2xy & yz^2\end{vmatrix}\\ &= (0-2yz)\boldsymbol{i} - (3x^2z - 0)\boldsymbol{j} + (0-2x)\boldsymbol{k}\\ &= -2yz\boldsymbol{i} - 3x^2z\boldsymbol{j} - 2x\boldsymbol{k}.\end{aligned}$$

(2) Giving $\boldsymbol{F} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$, curl of $\boldsymbol{F}$ is

$$\begin{aligned}
\operatorname{curl}\boldsymbol{F} &= \boldsymbol{\nabla}\times\boldsymbol{F} \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} \\
&= \boldsymbol{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yz & xy \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz & xz \end{vmatrix} \\
&= (x-x)\boldsymbol{i} - (y-y)\boldsymbol{j} + (z-z)\boldsymbol{k} \\
&= 0\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k} \\
&= \boldsymbol{0}.
\end{aligned}$$

This is a *irrotational vector field*, since curl of $\boldsymbol{F}$ is a zero vector; that is, $\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{0}$.

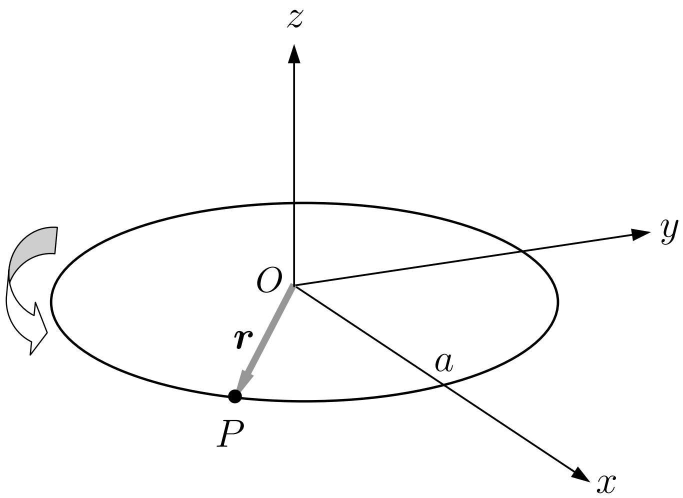
(3) Giving $\boldsymbol{F} = r^2\boldsymbol{r} = r^2x\boldsymbol{i} + r^2y\boldsymbol{j} + r^2z\boldsymbol{k}$, curl of $\boldsymbol{F}$ is

$$\begin{aligned}
\operatorname{curl}\boldsymbol{F} &= \boldsymbol{\nabla}\times\boldsymbol{F} \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^2x & r^2y & r^2z \end{vmatrix} \\
&= \boldsymbol{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^2y & r^2z \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ r^2x & r^2z \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ r^2x & r^2y \end{vmatrix} \\
&= (2yz - 2zy)\boldsymbol{i} - (2xz - 2zx)\boldsymbol{j} + (2xy - 2yx)\boldsymbol{k} \\
&= \boldsymbol{0}.
\end{aligned}$$

This $\boldsymbol{F}$ is also irrotational.

4.1 Physical Interpretation

The curl of a vector function $\boldsymbol{v}$ is related to the amount of rotation associated with $\boldsymbol{v}$. Consider a particle P rotating about O in the xy-plane at constant radius a and angular speed ω (rad/sec) as shown below,



The *angular velocity* of P is $\boldsymbol{\omega} = \omega\boldsymbol{k}$. The position of P is given by

$$\boldsymbol{r}(t) = \underbrace{a\cos(\omega t)}_{=x}\,\boldsymbol{i} + \underbrace{a\sin(\omega t)}_{=y}\,\boldsymbol{j} + \underbrace{0}_{=z}\,\boldsymbol{k}$$

from which we obtain the velocity of P as

$$\begin{aligned}\boldsymbol{v}(x,y,z) &= \frac{d\boldsymbol{r}}{dt}\\ &= -a\omega\sin(\omega t)\boldsymbol{i} + a\omega\cos(\omega t)\boldsymbol{j}\\ &= -\omega y\boldsymbol{i} + \omega x\boldsymbol{j}.\end{aligned}$$

Now, the curl of $\boldsymbol{v}$ is given by

$$\begin{aligned}\operatorname{curl}\boldsymbol{v} &= \boldsymbol{\nabla}\times\boldsymbol{v}\\ &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -\omega y & \omega x & 0\end{vmatrix}\\ &= \boldsymbol{i}\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \omega x & 0\end{vmatrix} - \boldsymbol{j}\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ -\omega y & 0\end{vmatrix} + \boldsymbol{k}\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ -\omega y & \omega x\end{vmatrix}\\ &= 0\boldsymbol{i} - 0\boldsymbol{j} + (\omega+\omega)\boldsymbol{k}\\ &= 2\omega\boldsymbol{k}\\ &= 2\boldsymbol{\omega}.\end{aligned}$$

The result, $\operatorname{curl} \boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{v} = 2\boldsymbol{\omega}$ indicates that $\operatorname{curl} \boldsymbol{v}$ is <u>twice</u> the angular velocity vector. In general, $\operatorname{curl} \boldsymbol{v}$ measures the amount of rotation or *vorticity* of $\boldsymbol{v}$. Hence, the term *irrotational* is used for vector fields having $\operatorname{curl} \boldsymbol{v} = \boldsymbol{0}$.

5 Identities Involving Divergence and Curl

Let $\boldsymbol{F}(x, y, z)$ and $\boldsymbol{G}(x, y, z)$ be vector functions, and $\phi(x, y, z)$ a scalar function. Assuming all of the derivatives implied below exist, we have the following indentities involving divergence and curl operations,

(1) $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} = 0\,;$

(2) $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \boldsymbol{0}\,;$

(3) $\boldsymbol{\nabla} \cdot (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{\nabla} \cdot \boldsymbol{G}\,;$

(4) $\boldsymbol{\nabla} \times (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla} \times \boldsymbol{G}\,;$

(5) $\boldsymbol{\nabla} \cdot \phi\boldsymbol{F} = \phi\boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \boldsymbol{\nabla}\phi\,;$

(6) $\boldsymbol{\nabla} \times \phi\boldsymbol{F} = \phi\boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla}\phi \times \boldsymbol{F}\,;$

(7) $\boldsymbol{\nabla} \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} - \boldsymbol{F} \cdot \boldsymbol{\nabla} \times \boldsymbol{G}\,;$

(8) $\boldsymbol{\nabla} \times (\boldsymbol{F} \times \boldsymbol{G}) = (\boldsymbol{G} \cdot \boldsymbol{\nabla})\boldsymbol{F} - \boldsymbol{G}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - (\boldsymbol{F} \cdot \boldsymbol{\nabla})\boldsymbol{G} + \boldsymbol{F}(\boldsymbol{\nabla} \cdot \boldsymbol{G})\,;$

(9) $\boldsymbol{\nabla}(\boldsymbol{F} \cdot \boldsymbol{G}) = (\boldsymbol{G} \cdot \boldsymbol{\nabla})\boldsymbol{F} + (\boldsymbol{F} \cdot \boldsymbol{\nabla})\boldsymbol{G} + \boldsymbol{G} \times (\boldsymbol{\nabla} \times \boldsymbol{F}) + \boldsymbol{F} \times (\boldsymbol{\nabla} \times \boldsymbol{G})\,;$

(10) $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \nabla^2\phi \quad \text{(by definition)}\,;$

(11) $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \nabla^2\boldsymbol{F}.$

Note that $\nabla^2 = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ is a scalar operator known as a *Laplacian operator*,

$$
\begin{aligned}
\nabla^2 &= \nabla \cdot \nabla \\
&= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \cdot \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \\
&= \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z} \\
&= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.
\end{aligned}
$$

EXAMPLE

Verify Identity (1): $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} = 0$ (divergence of the curl of a vector function is always zero).

SOLUTION

Let

$$\boldsymbol{F}(x, y, z) = F_1(x, y, z)\boldsymbol{i} + F_2(x, y, z)\boldsymbol{j} + F_3(x, y, z)\boldsymbol{k},$$

then

$$\begin{aligned}
\operatorname{curl} \boldsymbol{F} &= \boldsymbol{\nabla} \times \boldsymbol{F} \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \boldsymbol{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \boldsymbol{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \boldsymbol{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right), \\
\Rightarrow \operatorname{div} \operatorname{curl} \boldsymbol{F} &= \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \\
&= 0,
\end{aligned}$$

assuming pairs of mixed partial derivatives are equal.

EXAMPLE

Verify Identity (2): $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \boldsymbol{0}$ (curl of the gradient of a scalar function is always zero).

SOLUTION

If $\phi(x, y, z)$ is a scalar function, then

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x}\boldsymbol{i} + \frac{\partial \phi}{\partial y}\boldsymbol{j} + \frac{\partial \phi}{\partial z}\boldsymbol{k},$$

and

$$\begin{aligned}
\operatorname{curl} \boldsymbol{\nabla}\phi &= \boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}
\end{aligned}$$

$$= \boldsymbol{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \boldsymbol{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \boldsymbol{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$$
$$= \mathbf{0},$$

assuming pairs of mixed partial derivatives are equal. Thus, if $\boldsymbol{F}$ is the gradient of a scalar function, $\boldsymbol{F} = \boldsymbol{\nabla}\phi$, then $\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$ (irrotational).

6 Scalar Potential

6.1 Existence of a Scalar Potential

Suppose that $\boldsymbol{F}(x, y, z)$ is a vector function which is continuous in a simply connected region R in space. Then the following are equivalent (see Identity 2):

- $\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$ (irrotational) ;
- $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ for some scalar function $\phi(x, y, z)$.

That is, if

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0},$$

then there will exist a *scalar potential* function ϕ associated with $\boldsymbol{F}$, and $\boldsymbol{F}$ is said to be *conservative* or *irrotational* in the region R.

If a scalar potential for $\boldsymbol{F}$ exists, it means that the information contained in the three components of $\boldsymbol{F}$ actually derives from a single scalar function ϕ. This property leads to significant saving in computational cost for certain problems. For example, potential flows over an aircraft wing in subsonic, transonic and supersonic speed regime.

6.2 Finding the Scalar Potential

To find the scalar potential associated with $\boldsymbol{F}$, we must first check that

$$\operatorname{curl} \boldsymbol{F} = \boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0},$$

and then integrate the three equations represented by

$$\boldsymbol{\nabla}\phi = \boldsymbol{F}$$

to find a consistent solution for $\phi(x, y, z)$.

EXAMPLE

Verify that the vector field

$$\boldsymbol{F} = (2x + \sin y)\boldsymbol{i} + \big(x\cos y + z^2\big)\boldsymbol{j} + 2yz\boldsymbol{k}$$

is conservative, and find a scalar potential $\phi(x, y, z)$.

<u>SOLUTION</u>

Verifying $\operatorname{curl}\boldsymbol{F} = \boldsymbol{0}$:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + \sin y & x\cos y + z^2 & 2yz \end{vmatrix} \\
&= \boldsymbol{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\cos y + z^2 & 2yz \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2x + \sin y & 2yz \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2x + \sin y & x\cos y + z^2 \end{vmatrix} \\
&= (2z - 2z)\boldsymbol{i} - (0 - 0)\boldsymbol{j} + (\cos y - \cos y)\boldsymbol{k} \\
&= \boldsymbol{0}.
\end{aligned}$$

Since $\boldsymbol{F}$ is conservative, $\phi(x, y, z)$ exists. Now, equate $\boldsymbol{\nabla}\phi$ to $\boldsymbol{F}$:

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= 2x + \sin y, \\
\frac{\partial \phi}{\partial y} &= x\cos y + z^2, \\
\frac{\partial \phi}{\partial z} &= 2yz.
\end{aligned}$$

Integrating the above relations provides

$$\begin{aligned}
\phi &= x^2 + x\sin y + f(y, z), \\
\phi &= x\sin y + yz^2 + g(x, z), \\
\phi &= yz^2 + h(x, y),
\end{aligned}$$

where f, g and h are arbitrary functions. A <u>consistent solution</u> for ϕ cannot be obtained without suitably choosing these functions. Comparing the three equations, we can see that a consistent solution is

$$\phi(x, y, z) = x^2 + x\sin y + yz^2 + c,$$

where c is an arbitrary constant.

EXAMPLE

Verify that the vector field,

$$\boldsymbol{F} = (y+z)\boldsymbol{i} + (z+x)\boldsymbol{j} + (x+y)\boldsymbol{k},$$

is conservative, and find a scalar potential ϕ.

SOLUTION

The vector field $\boldsymbol{F}$ is conservative if $\operatorname{curl}\boldsymbol{F} = \mathbf{0}$:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} \\ &= (1-1)\boldsymbol{i} - (1-1)\boldsymbol{j} + (1-1)\boldsymbol{k} \\ &= \mathbf{0}.\end{aligned}$$

Equating $\boldsymbol{\nabla}\phi$ to $\boldsymbol{F}$:

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= y+z, \\ \frac{\partial\phi}{\partial y} &= z+x, \\ \frac{\partial\phi}{\partial z} &= x+y.\end{aligned}$$

There are two ways of proceeding from here.

METHOD 1:

As usual, we integrate to get

$$\begin{aligned}\phi &= xy + xz + f(y,z), \\ \phi &= yz + xy + g(x,z), \\ \phi &= xz + yz + h(x,y).\end{aligned}$$

By inspection, a consistent solution is

$$\phi(x,y,z) = xy + xz + yz + c,$$

where c is an arbitrary constant.

METHOD 2:

Integrating the first equation, $\frac{\partial\phi}{\partial x} = y+z$, gives

$$\phi = xy + xz + f(y,z),$$

where $f(y, z)$ is an arbitrary function not depending on the variable x. That is, $\frac{\partial f}{\partial x} = 0$. Differentiating this relation with respect to the variable y provides

$$\frac{\partial \phi}{\partial y} = x + \frac{\partial f}{\partial y}.$$

Comparing this equation with the second equation yields

$$\frac{\partial \phi}{\partial y} = z + x = x + \frac{\partial f}{\partial y} \qquad \Rightarrow z = \frac{\partial f}{\partial y},$$

which means that $f(y, z) = yz + g(z)$. Thus,

$$\phi = xy + xz + yz + g(z).$$

Differentiating this relation with respect to z gives

$$\frac{\partial \phi}{\partial z} = x + y + \frac{dg}{dz},$$

which on comparison with the third equation yields

$$\frac{\partial \phi}{\partial z} = x + y + \frac{dg}{dz} = x + y \qquad \Rightarrow \frac{dg}{dz} = 0.$$

That is,

$$g = c \quad \text{(constant)}.$$

Finally, the solution is

$$\phi(x, y, z) = xy + xz + yz + c.$$

6.3 Laplace's Equation

If $\boldsymbol{F}$ is a vector function which is both irrotational

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0} \quad \text{with } \boldsymbol{F} = \boldsymbol{\nabla}\phi,$$

and solenoidal,

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0,$$

then the associated scalar potential function ϕ satisfies

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{F} &= \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi \\ &= \nabla^2 \phi \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \\ &= 0. \end{aligned}$$

This is the famous *Laplace's equation*. Any function ϕ satisfies Laplace's equation is a *harmonic function*.

EXAMPLE

Find a scalar function $f(x,y,z)$ such that the vector function,

$$\boldsymbol{F} = (2z + 6xy)\boldsymbol{i} + f(x,y,z)\boldsymbol{j} + (2x - 6yz)\boldsymbol{k},$$

is both irrotational and solenoidal. Find a scalar potential $\phi(x,y,z)$ such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$.

<u>SOLUTION</u>

Requiring $\boldsymbol{F}$ to be solenoidal,

$$\operatorname{div}\boldsymbol{F} = 6y + \frac{\partial f}{\partial y} - 6y = \frac{\partial f}{\partial y} = 0.$$

Hence, f must be a function of x and z only. Next, requiring $\boldsymbol{F}$ to be irrotational,

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z + 6xy & f(x,z) & 2x - 6yz \end{vmatrix} \\ &= \boldsymbol{i}\left(-6z - \frac{\partial f}{\partial z}\right) - \boldsymbol{j}\big(2 - 2\big) + \boldsymbol{k}\left(\frac{\partial f}{\partial x} - 6x\right) \\ &= \boldsymbol{0}.\end{aligned}$$

Thus, $\dfrac{\partial f}{\partial x} = 6x$ and $\dfrac{\partial f}{\partial z} = -6z$. Integrating these two relations provides

$$\begin{aligned} f(x,z) &= 3x^2 + g(z), \\ f(x,z) &= -3z^2 + h(x). \end{aligned}$$

Thus, $f = 3x^2 - 3z^2 + c_0$, where c_0 is a constant. For convenience, we can set $c_0 = 0$, hence

$$\boldsymbol{F} = (2z + 6xy)\boldsymbol{i} + \big(3x^2 - 3z^2\big)\boldsymbol{j} + (2x - 6yz)\boldsymbol{k}.$$

Equating $\boldsymbol{\nabla}\phi$ to $\boldsymbol{F}$, we have that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= 2z + 6xy, \\ \frac{\partial \phi}{\partial y} &= 3x^2 - 3z^2, \\ \frac{\partial \phi}{\partial z} &= 2x - 6yz. \end{aligned}$$

Therefore,

$$\begin{aligned} \phi &= 2xz + 3x^2y + f_1(y,z), \\ \phi &= 3x^2y - 3yz^2 + f_2(x,z), \\ \phi &= 2xz - 3yz^2 + f_3(x,y). \end{aligned}$$

A consistent solution for ϕ is

$$\phi(x, y, z) = 2xz + 3x^2y - 3yz^2 + c,$$

where c is a constant.

6.4 Physical Interpretation

If $\boldsymbol{F}(x, y, z)$ represents a force field, then the scalar potential $\phi(x, y, z)$ has the dimensions of energy, and is related to the *potential energy* $V(x, y, z)$ by

$$V = -\phi.$$

For example, the force due to gravity acting on a body of mass m near to the earth's surface is governed by

$$\boldsymbol{F} = -mg\boldsymbol{k}.$$

Since $\boldsymbol{F}$ is constant, it is clear that curl $\boldsymbol{F} = \boldsymbol{0}$, so that a scalar potential ϕ exists. Equating $\boldsymbol{\nabla}\phi$ to $\boldsymbol{F}$, we have

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial z} = -mg.$$

A consistent solution is

$$\phi(x, y, z) = -mgz + \text{constant}.$$

Since the gravitational potential energy of such a body is given by $V = mgz + c$, then $V = -\phi$ (up to an additive constant).

7 Review Questions

[1] (a) Find the divergence and curl of the vector field,

$$\boldsymbol{V}(x,y,z) = 2xe^{yz}\boldsymbol{i} + \big(x^2ze^{yz} + 3y^2z\big)\boldsymbol{j} + \big(y^3 + x^2ye^{yz}\big)\boldsymbol{k}.$$

(b) Is $\boldsymbol{V}$ solenoidal? Is $\boldsymbol{V}$ irrotational? Would you expect a scalar potential function ϕ to exist for $\boldsymbol{V}$? Briefly explain each of your answers.

[2] (a) Verify that the following vector field,

$$\boldsymbol{F}(x,y,z) = 2x\sin(y+z)\boldsymbol{i} + \left(x^2\cos(y+z) + \frac{z^3}{\sqrt{y}}\right)\boldsymbol{j} + \big(x^2\cos(y+z) + 6z^2\sqrt{y}\big)\boldsymbol{k}$$

is irrotational.

(b) Find a scalar potential function for this field.

[3] (a) Verify that the vector field,

$$\boldsymbol{F}(x,y,z) = \left(\frac{y}{z} + x^2\right)\boldsymbol{i} + \left(\frac{x}{z} - \sin y\right)\boldsymbol{j} + \left(\cos z - \frac{xy}{z^2}\right)\boldsymbol{k},$$

is irrotational.

(b) Find a scalar potential function for this field.

[4] The vector field G is defined by

$$\boldsymbol{G}(x,y,z) = \big(3x^2 - 3y^2\big)\boldsymbol{i} + \big(12y^2z - 6xy - 4z^3\big)\boldsymbol{j} + \big(4y^3 - 12yz^2\big)\boldsymbol{k}.$$

(a) Determine $\boldsymbol{\nabla}\cdot\boldsymbol{G}$.

(b) Determine $\boldsymbol{\nabla}\times\boldsymbol{G}$.

(c) Is $\boldsymbol{G}$ solenoidal? is $\boldsymbol{G}$ irrotational?

[5] Determine the unit vector which is normal to the surface,

$$z = 5 - \sqrt{x^2 + y^2},$$

at the point $P(4,3,0)$ and is directed *away* from the origin.

[6] Find the directional derivative of $\phi = z^4 + x^2y^3$ at the point $(2,-1,1)$ in the direction $6\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$.

[7] Consider the vector field $\boldsymbol{F}$ defined by

$$\boldsymbol{F} = \big(1 + 2xyz^3\big)\boldsymbol{i} + \big(2y + x^2z^3\big)\boldsymbol{j} + \big(\alpha x^2yz^2\big)\boldsymbol{k}.$$

(a) Determine the value of α for which $\boldsymbol{F}$ is a conservative field.

(b) Assuming α takes the value found in part (a), determine a scalar potential ϕ for $\boldsymbol{F}$; that is, determine a scalar field ϕ such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$.

[8] If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, so that $|\boldsymbol{r}| = r = \sqrt{x^2 + y^2 + z^2}$:

(a) Find $\dfrac{\partial r}{\partial x}$, and write the answer in terms of r and x. Hence, write down

$$\frac{\partial r}{\partial y} \quad \text{and} \quad \frac{\partial r}{\partial z}.$$

(b) Using the results obtained in part (a), or otherwise, evaluate $\operatorname{div}(\boldsymbol{r}/r)$.

[9] Find a unit normal vector to the paraboloid defined by $z(x, y) = 4x^2 + y^2$ at the point $(2, 3, 25)$.

[10] The vector field $\boldsymbol{G}$ and the scalar field ϕ are defined by

$$\boldsymbol{G} = \left(4zy^2 + 2x - 5\right)\boldsymbol{i} + \left(2z^2 - 3x + y\right)^2\boldsymbol{j} + \left(3xy + 2z\right)\boldsymbol{k}$$

and

$$\phi(x, y, z) = xz^2 - 4xy^2,$$

respectively. Compute

(a) $\boldsymbol{\nabla} \cdot \boldsymbol{G}$;

(b) $\boldsymbol{\nabla} \times \boldsymbol{G}$;

(c) $\boldsymbol{\nabla}\phi$;

(d) $\operatorname{curl}\operatorname{grad}\phi$.

[11] Consider the vector field $\boldsymbol{H}$ defined by

$$\boldsymbol{H} = (2 + 2xyz)\boldsymbol{i} + \left(4 + x^2z\right)\boldsymbol{j} + \left(2z + x^2y\right)\boldsymbol{k}.$$

(a) Show that $\boldsymbol{H}$ is a conservative field.

(b) Determine a scalar potential $\phi(x, y, z)$ for $\boldsymbol{H}$; that is, determine a scalar field ϕ such that $\boldsymbol{H} = \boldsymbol{\nabla}\phi$.

(c) Determine $\operatorname{curl}(\phi\boldsymbol{H})$, where $\boldsymbol{H}$ and ϕ are as above.
Hint: Use the identity,

$$\operatorname{curl}\left(\phi\boldsymbol{H}\right) = \phi\boldsymbol{\nabla} \times \boldsymbol{H} + \boldsymbol{\nabla}\phi \times \boldsymbol{H}.$$

[12] If $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, so that $r = \sqrt{x^2 + y^2 + z^2}$:

(a) Show that $\nabla r^4 = 4r^2\boldsymbol{r}$;

(b) Using the identity,

$$\mathrm{div}\,(\phi\boldsymbol{F}) = \phi\nabla\cdot\boldsymbol{F} + \nabla\phi\cdot\boldsymbol{F},$$

together with the result from part (a), show that

$$\mathrm{div}\left(r^4\boldsymbol{r}\right) = 7r^4.$$

[13] A vector field $\boldsymbol{A}$ is said to be a *vector potential* for a vector field $\boldsymbol{V}$ if

$$\boldsymbol{V} = \nabla\times\boldsymbol{A}.$$

Verify that the vector field,

$$\boldsymbol{A}(x,y,z) = x^2y^2e^z\boldsymbol{i} + xye^z\boldsymbol{j} + x^2y^3z^4\boldsymbol{k}$$

is a vector potential for the vector field,

$$\boldsymbol{V}(x,y,z) = \left(3x^2y^2z^4 - xye^z\right)\boldsymbol{i} + \left(x^2y^2e^z - 2xy^3z^4\right)\boldsymbol{j} + \left(ye^z - 2x^2ye^z\right)\boldsymbol{k}.$$

8 Answers to Review Questions

[1] (a) $\operatorname{div}\boldsymbol{V} = \boldsymbol{\nabla}\cdot\boldsymbol{V} = 2e^{yz} + \big(x^2z^2e^{yz} + 6yz\big) + x^2y^2e^{yz}$.
$\operatorname{curl}\boldsymbol{V} = \boldsymbol{\nabla}\times\boldsymbol{V} = \big(3y^2 + x^2e^{yz}(1+yz)\big)\boldsymbol{i}$.

(b) $\boldsymbol{V}$ is not solenoidal, since $\operatorname{div}\boldsymbol{V} \neq 0$. $\boldsymbol{V}$ is not irrotational, since $\operatorname{curl}\boldsymbol{V} \neq \boldsymbol{0}$. Since $\operatorname{curl}\boldsymbol{V}$ is not zero, a scalar potential function $\phi(x,y,z)$ will not exist.

[2] (a) $\operatorname{curl}\boldsymbol{F}$:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{F} &= \left(-x^2\sin(y+z) + \frac{3z^2}{\sqrt{y}} + x^2\sin(y+z) - \frac{3z^2}{\sqrt{y}}\right)\boldsymbol{i}\\ &- \big(2x\cos(y+z) - 2x\cos(y+z)\big)\boldsymbol{j} + \big(2x\cos(y+z) - 2x\cos(y+z)\big)\boldsymbol{k}\\ &= \boldsymbol{0}.\end{aligned}$$

(b) $\phi(x,y,z) = x^2\sin(y+z) + 2z^3\sqrt{y} + c$, where c is a constant.

[3] (a) $\boldsymbol{\nabla}\boldsymbol{F} = \left(-\frac{x}{z^2} + \frac{x}{z^2}\right)\boldsymbol{i} - \left(-\frac{y}{z^2} + \frac{y}{z^2}\right)\boldsymbol{j} + \left(\frac{1}{z} - \frac{1}{z}\right)\boldsymbol{k} = \boldsymbol{0}$

(b) $\phi(x,y,z) = \frac{xy}{z} + \frac{x^3}{3} + \cos y + \sin z + c$, where c is a constant.

[4] (a) $\boldsymbol{\nabla}\cdot\boldsymbol{G} = 0$

(b) $\boldsymbol{\nabla}\times\boldsymbol{G} = \boldsymbol{0}$

(c) $\boldsymbol{G}$ is solenoidal and irrotational.

[5] Let $F(x,y,z) = x^2 + y^2 - z^2 + 10z - 25$. Normal vector is

$$\begin{aligned}\boldsymbol{n} &= \boldsymbol{\nabla}F\\ &= 2x\boldsymbol{i} + 2y\boldsymbol{j} + (10-2z)\boldsymbol{k}\\ &= 8\boldsymbol{i} + 6\boldsymbol{j} + 10\boldsymbol{k} \quad \text{at point } P.\end{aligned}$$

Thus, $\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} = \frac{1}{\sqrt{50}}(4\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k})$.

[6] $\boldsymbol{\nabla}\phi = 2xy^3\boldsymbol{i} + 3x^2y^2\boldsymbol{j} + 4z^3\boldsymbol{k} = -4\boldsymbol{i} + 12\boldsymbol{j} + 4\boldsymbol{k}$ at point $(2,-1,1)$.
Directional derivative is $\boldsymbol{\nabla}\phi\cdot\hat{\boldsymbol{u}} = 12/7$.

[7] (a) Curl of $\boldsymbol{F}$:
$\boldsymbol{\nabla}\times\boldsymbol{F} = \big(\alpha x^2z^2 - 3x^2z^2\big)\boldsymbol{i} - \big(2\alpha xyz^2 - 6xyz^2\big)\boldsymbol{j} + \big(2xz^3 - 2xz^3\big)\boldsymbol{k} = \boldsymbol{0}$ if $\alpha = 3$.

(b) $\frac{\partial\phi}{\partial x} = 1 + 2xyz^3;\quad \frac{\partial\phi}{\partial y} = 2y + x^2z^3;\quad \frac{\partial\phi}{\partial z} = 3x^2yz^2.$
Scalar potential function, $\phi(x,y,z) = x + y^2 + x^2yz^3 + c$, where c is a constant.

[8] (a) $\frac{\partial r}{\partial x} = \frac{x}{r};\quad \frac{\partial r}{\partial y} = \frac{y}{r};\quad \frac{\partial r}{\partial z} = \frac{z}{r}.$

(b) Divergence of $\boldsymbol{r}/r$:

$$\begin{aligned}\operatorname{div}(\boldsymbol{r}/r) &= \boldsymbol{\nabla}\cdot\left(\frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{r}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)\\ &= \frac{r^2-x^2}{r^3}+\frac{r^2-y^2}{r^3}+\frac{r^2-z^2}{r^3}\\ &= \frac{3r^2-r^2}{r^3}\\ &= \frac{2}{r}\,.\end{aligned}$$

[9] Let $\phi = 4x^2+y^2-z$.
Normal vector: $\boldsymbol{n} = \boldsymbol{\nabla}\phi = 16\boldsymbol{i}+6\boldsymbol{j}-\boldsymbol{k}$ at point $(2,3,25)$.
Unit normal vector: $\hat{\boldsymbol{n}} = \pm\dfrac{(16,6,-1)}{\sqrt{293}}$.

[10] (a) $\boldsymbol{\nabla}\cdot\boldsymbol{G} = 4+2\big(2z^2-3x+y\big)$

(b) $\boldsymbol{\nabla}\times\boldsymbol{G} = \big(3x-16z^3+24xz-8yz\big)\boldsymbol{i}-\big(3y-4y^2\big)\boldsymbol{j}+\big(-12z^2+18x-6y-8yz\big)\boldsymbol{k}$

(c) $\boldsymbol{\nabla}\phi = \big(z^2-4y^2\big)\boldsymbol{i}+\big(-8xy\big)\boldsymbol{j}+\big(2xz\big)\boldsymbol{k}$

(d) $\operatorname{curl}\operatorname{grad}\phi = \boldsymbol{\nabla}\times\boldsymbol{\nabla} = \boldsymbol{0}$

[11] (a) $\boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{0}$

(b) $\dfrac{\partial\phi}{\partial x} = 2+2xyz\,;\quad \dfrac{\partial\phi}{\partial y} = 4+x^2z\,;\quad \dfrac{\partial\phi}{\partial z} = 2z+x^2y.$

Scalar potential function: $\phi(x,y,z) = 2x+4y+z^2+x^2yz+c$, where c is a constant.

(c) $\operatorname{curl}\big(\phi\boldsymbol{H}\big) = \phi\boldsymbol{\nabla}\times\boldsymbol{H}+\boldsymbol{\nabla}\phi\times\boldsymbol{H} = \phi\,\boldsymbol{0}+\boldsymbol{H}\times\boldsymbol{H} = \boldsymbol{0}$.

[12] (a) $\boldsymbol{\nabla}r^4 = 4r^2x\boldsymbol{i}+4r^2y\boldsymbol{j}+4r^2z\boldsymbol{k} = 4r^2\boldsymbol{r}$.

(b) $\operatorname{div}\big(r^4\boldsymbol{r}\big) = r^4\boldsymbol{\nabla}\cdot\boldsymbol{r}+\boldsymbol{\nabla}r^4\cdot\boldsymbol{r} = 3r^4+4r^2\boldsymbol{r}\cdot\boldsymbol{r} = 7r^4$.

[13] Not available.